Chapter 3

Schemes: A Closer Study

In this chapter, we take a closer look at some elementary properties of schemes.

Section 3.1 Dimension

Definition 3.1.1. A topological space is called *noetherian* if every descending chain of closed subsets terminates.

Remark 3.1.2. If R is anoetherian ring, then $X = Spec(R)$ is a noetherian topological space, since the closed subsets correspond to ideals.

Proposition 3.1.3: *Every closed subset in a noetherian topological space can be uniquely decomposed as an irredundant union of irreducible subspaces.*

Proof: Use the same proof that showed the corresponding result for algebraic subsets of affine space over a field. ∎

Definition 3.1.4. Let X be a noetherian topological space. The dimension of X is defined to be

$$dim(X) = sup\{n : \text{there exists a chain of nonempty irreducible distinct closed subsets } Z_0 \subset Z_1 \subset \cdots \subset Z_n \text{in} X\},$$

provided that this supremum exists.

Examples.

(i) $dim(*) = 0$.
(ii) If k is a field, then $dim(\mathbf{A}^1_k) = 1$, since the irreducible sets are either points or all of $\mathbf{A}^1$.
(iii) If k is a field, then $dim(\mathbf{A}^2_k) = 2$, since the irreducible sets are either all of $\mathbf{A}^2$, an irreducible curve, or a point.
(iv) With the usual topology on the real line $\mathbf{R}$, one has $dim(\mathbf{R}) = 0$. For $\mathbf{R}$ is not irreducible, and the only irreducible closed subsets of $\mathbf{R}$ are single points.
(v) $dim(Spec(\mathbf{Z})) = 1$, since every prime ideal is either zero or maximal.

Definition 3.1.5. Let P be a prime ideal in a ring R (as always, commutative with 1). The height of P is defined to be

$$ht(P) = sup\{n : \text{there exists a chain of distinct prime ideals } P_0 \subset P_1 \subset \cdots \subset P_n = P \subset R\}$$

Definition 3.1.6. The *Krull dimension* of a ring R is defined to be

$$dim(R) = sup\{ht(P) : P \subset R \text{is prime}\}.$$

Lemma 3.1.7: *Let $X = Spec(R)$ be an affine scheme. Then*

$$dim(X) = dim(R).$$

Proof: There is a one-to-one correspondence between the irreducible closed subsets of X and the prime ideals of R. ∎

Proposition 3.1.8: *Let X be a noetherian topological space. Then*

(i) *$dim(X) = sup(dim(X_i))$, where the X_i range through the irreducible components of X.*
(ii) *If $U \subset X$ is a dense open set and if $dim(U)$ is finite, then $dim(X) = dim(U)$.*
(iii) *If $\bar{Y} \subset X$, then $dim(Y) \leq dim(X)$. Moreover, if X is irreducible and the inclusion is proper, then the inequality is strict.*

Proof: (i) Let $Z_0 \subset \ldots \subset Z_n$ be a proper chain of irreducible closed subsets of X. Since

$$Z_n = \bigcup_i (X_i \cap Z_n)$$

and Z_n is irreducible, there exists an i such that $Z_n \subset X_i$. But then the entire chain is contained inside X_i.

(ii) By (i), we may assume that X is irreducible. If $Z_0 \subset \ldots \subset Z_n$ is a proper chain of closed irreducible subsets of U, then their closures $\bar{Z}_0 \subset \cdots \subset \bar{Z}_n$ form a proper chain of closed irreducibles of X. (Properness follows because $Z_i = U \cap \bar{Z}_i$, and closures always stay irreducible.) So $dim(U) \leq dim(X)$.

Since $dim(U)$ is finite, we can choose a chain $Z_0 \subset \cdots Z_n$ in U of maximal length. Now let W be any irreducible closed subset of X such that $\bar{Z}_i \subset W \subset \bar{Z}_{i+1}$. Intersecting back with U yields

$$Z_i \subset W \cap U \subset Z_{i+1}.$$

Since the original chain had maximal length, one of these inclusions must be an equality. But $W \cap U$ is dense in the irreducible space W, hence one of the earlier inclusions in X was already an equality. It follows that $\bar{Z}_0 \subset \cdots \subset \bar{Z}_n$ is a maximal chain in X, and the dimensions are equal.

(iii) Without loss of generality, we may assume that $Y = \bar{Y}$. Let $Z_0 \subset \cdots \subset Z_n$ be a proper chain in Y. Then it is also a proper chain in X. If the containment is proper, then we can add X to the top of the chain to lengthen it. ∎

Proposition 3.1.9: *Let X be a scheme that is a noetherian topological space. Then $dim(X) = 0$ if and only if X is a finite set with the discrete topology.*

Proof: It is clear that any set with the discrete topology has dimension zero, since the only irreducible subsets are singeltons. On the other hand, we know that every scheme contains at least one closed point (coming from a maximal ideal inside some ring defining an open affine subset). If X contained a non-closed point, then its closure would be irreducible, forcing the dimension of X to be greater than zero. So, any zero-dimensional scheme has the property that every point is closed. Thus, in any affine open subset $Spec(R)$ of X, the ring R must have the property that every prime ideal is maximal. This forces $Spec(R)$, and hence X to have the discrete topology. Now use the finite decomposition into irreducibles for the noetherian topological space X to obtain the desired conclusion. ∎

Theorem 3.1.10: *(Dimension Theorem) Let X be an algebraic variety over a field k. Then*

$$dim(X) = tr.deg_k(k(X)).$$

Proof: Since every variety contains an open, dense, affine variety, the result will follow if we can establish the following three facts.

(i) The coordinate ring $A(X)$ is an integral extension of a polynomial ring.
(ii) If $B \subset A$ is an integral extension of noetherian rings, then $dim(B) = dim(A)$ and $tr.deg(B) = tr.deg(A)$.
(iii) If k is a field, then $dim(\mathbf{A}_k^n) = n$.

∎

Let's start with the easiest one first.

Proposition 3.1.11: *If k is a field, then $dim(\mathbf{A}^n_k) = n$.*

Proof: Since $(0) \subset (x_1) \subset (x_1, x_2) \subset \cdots \subset (x_1, \ldots, x_n)$ is a proper chain of prime ideals, we have $dim(\mathbf{A}^n) \geq n$.

On the other hand, we know that $n = tr.deg(k[x_1, \ldots, x_n])$. Moreover, if A is an integral domain and if P is any nonzero prime ideal in A, then $tr.deg(A/P) < tr.deg(A)$. For, take any algebraically independent set $\bar{x}_1, \ldots, \bar{x}_r \in A/P$. Lift them arbitrarily to elements $x_1, \ldots, x_r \in A$, and choose a nonzero element $x \in P$. Then $\{x_1, \ldots, x_r, x\}$ is an algebraically independent set in A. To see this, suppose $F(t_1, \ldots, t_r, t_{r+1}) \in k[t_1, \ldots, t_{r+1}]$ is any polynomial such that $F(x_1, \ldots, x_r, x) = 0$ in A. Since A is an integral domain, we can assume that F is an irreducible polynomial. Then $F(\bar{x}_1, \ldots, \bar{x}_r, 0) = 0$ in A/P. Since there are no nonzero algebraic relations between these elements in A/P, the polynomial $F(t_1, \ldots, t_r, 0)$ must be identically zero. But then F is divisible by t_{r+1}; by irreducibility, $F = t_{r+1}$. Since $x \neq 0$ in A; this is a contradiction.

Now let $P_0 \subset \cdots \subset P_t$ be a chain of prime ideals in $k[x_1, \ldots, x_n]$. Then

$$n > tr.deg(A/P_0) > tr.deg(A/P_1) > \cdots > tr.deg(A/P_t),$$

so $n \geq t$. Thus, $dim(\mathbf{A}^n) \leq n$, and the result follows. ∎

In order to establish the other two parts, we need some preliminaries from commutative algebra; in particular, we need to know how prime ideals behave in integral extensions of rings. To understand that, we introduce one of the essential tools of commutative algebra: localization.

Definition 3.1.12. A ring is called *local* if it contains exactly one maximal ideal.

Definition 3.1.13. If A is a ring and if P is a prime ideal in A, we define the *localization* of A at P to be the ring A_P obtained by inverting all elements of $A \backslash P$.

Remark 3.1.14. The localization A_P is a local ring, with maximal ideal PA_P.

Definition 3.1.15. Let $f : B \subset A$ be an extension of rings. If $P \subset A$ is a prime ideal, then $Q = f^{-1}(P) = P \cap B$ is a prime ideal in B. In this circumstance, we say that P *lies over* Q.

Lemma 3.1.16: *Let B be a local ring with maximal ideal Q. Let $B \subset A$ be an integral extension. Then the set of prime ideals of A lying over Q is just the set of maximal ideals of A.*

Proof: We show first that every maximal ideal M of A lies over Q. Define $N = M \cap B$. Then $\bar{A} = A/M$ is a field that is integral over the subring $\bar{B} = B/N$. Now, let $0 \neq x \in \bar{B}$. Since $1/x \in \bar{A}$, we have a monic polynomial equation

$$(1/x)^n + b_1(1/x)^{n-1} + \cdots + b_n = 0$$

with coefficients in $\bar{B}$. Now multiply by x^{n-1} to get

$$(1/x) = -(b_1 + b_2 x + \cdots + b_n x^{n-1}) \in \bar{B}.$$

So, $\bar{B}$ is a field and $N = Q$ is the unique maximal ideal in B.

Next, we must show that every ideal P in A that lies over Q is maximal. This time, $\bar{A} = A/P$ is an integral domain that is integral (thus algebraic) over the field $\bar{B} = B/Q$. Take $0 \neq y \in \bar{A}$. There is an algebraic dependence relation

$$b_0 y^n + b_1 y^{n-1} + \cdots + b_n = 0$$

of minimal degree with coefficients in $\bar{B}$. By minimality, $b_n \neq 0$. Since $\bar{B}$ is a field, we can divide the polynomial by b_n, and assume that $b_n = 1$. Now we have

$$\frac{1}{y} = -(b_0 y^{n-1} + b_1 y_{n-2} + \cdots + b_{n-1}) \in \bar{A}.$$

Hence, $\bar{A}$ is a field and P is maximal. ∎

Proposition 3.1.17: *(The Lying-Over Theorem) Let $B \subset A$ be an integral extension of rings. If $Q \subset B$ is any prime ideal, then there exists a prime ideal $P \subset A$ lying over Q.*

Proof: Let B_Q be the localization of B at Q. Then $A_Q = (B \backslash Q)^{-1} A$ is an integral extension of B_Q, and contains it as a subring. The prime ideals of A lying over Q correspond to the prime ideals of A_Q lying over QB_Q, which are necessarily the maximal ideals of A_Q. Since $B_Q \neq 0$, we know that A_Q is nonzero, so it has maximal ideals. ∎

Proposition 3.1.18: *(The Going-Up Theorem) Let $B \subset A$ be an integral extension of rings. Let $Q \subset Q' \subset B$ be prime ideals and let $P \subset A$ be a prime ideal lying over Q. Then there exists an ideal P' lying over Q' such that $P \subset P'$.*

Proof: The quotient A/P contains and is integral over B/Q. By the Lying Over Theorem, there exists a prime P'/P in A/P lying over the prime ideal Q'/Q. By the isomorphism theorems, P' is a prime ideal of A lying over Q'. ∎

Lemma 3.1.19: *Let B be an integral domain that is integrally closed in its field of fractions L. Let K/L be a normal extension with Galois group G, and let A be the integral closure of B in K. If Q is any prime ideal of B, then G acts transitively on the set of prime ideals lying over Q.*

Proof: One can assume (with some work that I'm omitting) that K/L is a finite Galois extension. Suppose now that P' and P are two primes lying over Q, and that P' is not contained in any of the conjugates $P_i = \sigma_i(P)$ of P for $\sigma_i \in G$. Then there is an element $x \in P'$ that is contained in no P_i. But $y = N_{K/L}(x) \in B$ is not in Q (because all $\sigma_i(x) \notin P$) and is in $P' \cap A$. This is a contradiction. ∎

Proposition 3.1.20: *(The Going-Down Theorem) Let $B \subset A$ be an integral extension of integral domains, and assume that B is integrally closed in its field of fractions. Let $Q \subset Q' \subset B$ be prime ideals and let $P' \subset A$ be a prime ideal lying over Q'. Then there exists an ideal P lying over Q such that $P \subset P'$.*

Proof: Let K be the field of fractions of A, let L be the field of fractions of B, and let F be the Galois closure of the field extension K/L. Inside F, let C denote the integral closure of B (and hence also of A). By the Lying Over Theorem, there exists a prime ideal Q_0 in C lying over Q. By the Going Up Theorem, there exists a prime ideal $Q_0' \supset Q_0$ lying over Q'. Also by the Lying Over Theorem, there exists a prime ideal $Q_1' \subset C$ lying over P'. Now the ideals Q_0' and Q_1' lie over the same ideal Q' in B. Because the extension is Galois, there exists an automorphism $\sigma \in Gal(F/L)$ such that $\sigma(Q_0') = Q_1'$. Now the result follows by taking $P = \sigma(Q_0) \cap A$. ∎

Now we can carry out part (ii) of the proof of the dimension theorem.

Proposition 3.1.21: *Let $B \subset A$ be an integral extension of noetherian rings. Then $dim(B) = dim(A)$ and $tr.deg(B) = tr.deg(A)$.*

Proof: Since integral extensions are algebraic, the equality for transcendence degrees is trivial.

The Lying Over and Going Up Theorems show that any chain of prime ideals in B can be lifted to a chain in A of the same length; thus, $dim(A) \geq dim(B)$. Conversely, if $P \subset P'$ are distinct prime ideals in A, then the ideals $P \cap B \subset P' \cap B$ must also be distinct. (Localize at the prime ideal $P \cap B$; since the ideals lying over its maximal ideal are precisely the maximal ideals of the localization of A, there cannot be any proper containments between them.) Thus, any chain of primes in A produces a chain of the same length in B, and $dim(A) \leq dim(B)$. ∎

Finally, we can carry out step (i).

Theorem 3.1.22: *Let $A = k[x_1, \dots, x_n]$ be a polynomial ring over a field k, and let P be an ideal of A of height h. Then there exist elements $v_1, \dots, v_n \in A$ such that*

(i) A is integral over $k[v]$, and

(ii) $P \cap k[v] = (v_1, \dots, v_h)$.

Proof: The proof is by induction on the height $h = ht(P)$. The case $h = 0$ is trivial, for then $P = 0$ and we can take $v_i = x_i$. A better base for the induction, however, is provided by the case $h = 1$. Choose a nonzero polynomial $v_1 = f(x) \in P$. Write $f(x) = \sum_{i=1}^{s} c_i M_i(x)$ where $c_i \in k$ and each $M_i(x)$ is a monomial. Given any positive integers $d_1, \dots, d_n$, we define the d-weight of a monomial $M(x) = \prod x_i^{a_i}$ to be $\sum a_i d_i$. Now choose weights $d_1 = 1, d_2, \dots, d_n$ so that all the monomials appearing in $f(x)$ have distinct weights. (We can achieve this by using large prime powers for the weights.) Put $v_i = x_i - x_1^{d_i}$ for $i = 2, \dots, n$. Then

$$\begin{aligned} v_1 = f(x) &= f(x_1, v_2 + x_1^{d_2}, \dots, v_n + x_1^{d_n}) \\ &= a_j x_1^e + g(x_1, v_2, \dots, v_n) \end{aligned}$$

where g is a polynomial whose degree in x_1 is strictly less than e, and a_j is the coefficient of the term of largest weight in f. It follows that x_1 is integral over the ring $k[v_1, \dots, v_n]$, and hence so are the elements $x_i = v_i + x_1^{d_i}$.

We're not done yet, however, since we only know that $k[v] \subset k[x]$ is an integral extension. We still need to mod out the prime ideal P and its pullback $P \cap k[v]$. We clearly have a containment $(v_1) \subset P \cap k[v]$. In fact, however, both these ideals are primes of height 1, so they must be equal. (Note: that $P \cap k[v]$ is height 1 uses the fact that $k[v]$ is integrally closed, so the Going Down Theorem applies.)

Now suppose $h > 1$. Choose an ideal $Q \subset P$ of height $h-1$. By induction, there exist $w_1, \dots, w_n$ such that $k[x]$ is integral over $k[w]$ and $Q \cap k[w] = (w_1, \dots, w_{h-1})$. Define $P' = P \cap k[w]$. Then P' also has height h, so it contains $(w_1, \dots, w_{h-1})$ properly. Choose a nonzero polynomial of the form $f(w_h, \dots, w_n) \in P'$ and repeat the weight argument of the case $h = 1$. ∎

Theorem 3.1.23: *(Noether Normalization Theorem) Let A be an integral domain that is finitely generated as an algebra over a field k. Then there exist elements $y_1, \dots, y_r \in A$ that are algebraically independent over k such that A is integral over $k[y_1, \dots, y_r]$.*

Proof: Write $A = k[x_1, \dots, x_n]/P$ where P is a prime ideal of height $n - r$. By the previous theorem, there are elements $v_1, \dots, v_n$ in $k[x_1, \dots, x_n]$ such that $k[v] \subset k[x]$ is integral and such that $P \cap k[v] = (v_{r+1}, \dots, v_n)$. The result follows by taking $y_i \equiv v_i \bmod P$. ∎

Corollary 3.1.24: *Let A be an integral domain that is finitely generated as a k-algebra. Then for any prime P in A, we have*

$$ht(P) + dim(A/P) = dim(A).$$

Proof: Reduce to the case $A = k[x_1, \dots, x_n]$ and use the proof of the Noether Normalization Theorem. ∎

Corollary 3.1.25: *If X and Y are algebraic varieties, then $dim(X \times Y) = dim(X) + dim(Y)$.*

Proof: The transcendence degree of a tensor product satisfies the corresponding relation. ∎

Proposition 3.1.26: *Let X be a hypersurface in $\mathbf{A}_k^n$. Then every irreducible component of X has dimension $n-1$.*

Proof: We may assume that X is irreducible, and hence of the form $Z(f)$ for a nonconstant irreducible polynomial $f \in k[x_1, \ldots, x_n]$. By renumbering the variables, we can assume that x_n actually occurs in f. Now let $t_i \in A(X)$ be the image of x_i. We claim that $\{t_1, \ldots, t_{n-1}\}$ is an algebraically independent set, and therefore $dim(X) \geq n-1$. For, suppose that $G(t_1, \ldots, t_{n-1}) = 0$ is an algebraic relation. Then $G \in I(X) = (f)$, so f divides G. But this is impossible, since x_n occurs in f but not in G. On the other hand, the proof that $\mathbf{A}^n$ has dimension n shows that $dim(X) \leq n-1$. ∎

Theorem 3.1.27: *Let $X \subset \mathbf{A}_k^n$ be an algebraic set all of whose irreducible components have dimension $n-1$. Then X is a hypersurface.*

Proof: Without loss of generality, we may assume that X is irreducible. Let $f \in I(X)$ be a nonzero polynomial. By irreducibility, some irreducible factor h of f must vanish on X. So, $X \subset Z(h)$. But this is an inclusion of irreducible sets of the same dimension, so it must be an equality. ∎

Section 3.2 Localizations

Definition 3.2.1. Let X be a scheme, and let $x \in X$ be a point. The stalk $\mathcal{O}_{X,x}$. is called the *local ring of X at x* .

Remark 3.2.2. By construction, the local ring $\mathcal{O}_{X,x}$ has a unique maximal ideal, which will be denoted m_x.

Definition 3.2.3. If x is a point of a scheme X, we define the *residue field* at P to be $k(x) = \mathcal{O}_{X,x}/m_x$

Remark 3.2.4. As we have seen, if $X = Spec(R)$ and if $x \in X$ is the point corresponding to the prime ideal $P \subset R$, then $\mathcal{O}_{X,x} = R_P$ and $m_x = PR_P$. In addition, even though the integral domain R/P need not be a field, its field of fractions is equal to $\mathcal{O}_{X,x} = R_P/PR_P$.

Proposition 3.2.5: *Let R be a noetherian ring and let P in R be a prime ideal. Then the local ring R_P is also noetherian.*

Proof: If I is an ideal in R_P, write $\bar{I} = I \cap R$ for its pullback to R. Since R is noetherian, the ideal $\bar{I}$ is finitely generated by some elements $a_1, \ldots, a_r$. Now consider the ideal J in R_P generated by the same elements. One has

$$J = (a_1, \ldots, a_r) \subset I.$$

Now suppose $x \in I$. Write $x = a/b$ with a, $b \in R$ and $b \notin P$. Since $a = bx \in I \cap R$, we have $a \in \bar{I}$, and so $a/b \in J$. In particular, the same set of generators works. ∎

Corollary 3.2.6: *Let P be a prime ideal in a noetherian ring R, and let $m = PR_P$ be the maximal ideal in the localization. Then the natural map $R \to R_P$ induces an isomorphism of $k(P)$-vector spaces $P/P^2 \otimes_{R/P} k(P) \to m/m^2$.*

Proof: This follows from the fact that we can take the same generating sets for the ideals P and m. ∎

Definition 3.2.7. Let x be a point in a scheme X, with local ring $\mathcal{O}_{X,x}$ and maximal ideal $m = m_x$. The *Zariski tangent space* to X at x is defined to be the dual $k(x)$-vector space

$$\Theta_{X,x} = Hom_{k(x)}(m/m^2, k(x)).$$

Remark 3.2.8. We have seen previously that another description of this vector space is as the set of morphisms $Mor(T, X; x)$, where T is the structured algebraic set with coordinate ring $k[\varepsilon]/\varepsilon^2$.

Example 3.2.9. Let's start with a concrete description of the tangent space to $\mathbf{A}^n$ at the origin. The corresponding maximal ideal in the polynomial ring $k[x_1, \ldots, x_n]$ is $M = (x_1, \ldots, x_n)$. So, the images of $x_1, \ldots, x_n$ form a basis of M/M^2. Let D_i be elements of the dual basis in $\Theta = Hom(M/M^2, k)$. As abstract algebraic objects, the D_i are characterized by the relation $D_i(x_j) = \delta_{ij}$, the Kronecker delta. Now a general tangent vector is a linear operator on M/M^2 that has the form $\Theta = \sum_{i=1}^n t_i D_i$ for an arbitrary choice of $t_i \in k$. In this way, we can identify the Θ with a copy of affine n-space. Moreover, we can give D_i a more natural interpretation. Let $F \in M$ be any function that vanishes at the origin. It is easy to see that

$$D_i(F) = \frac{\partial F}{\partial x_i}(0).$$

Now let $\Theta = \sum_{i=1}^n t_i D_i$ be an arbitrary tangent vector. If $F \in M$, then

$$\Theta(F) = \sum_{i=1}^n t_i \frac{\partial F}{\partial x_i}(0).$$

After identifying the tangent space with $\mathbf{A}^n$, the previous equation allows us to interpret F as a linear function on Θ, and hence as an element of the dual vector space $\Theta^* = Hom(\Theta, k)$.

Now consider the case of an algebraic set $X \in \mathbf{A}^n$, with $x = (0, \ldots, 0)$. Let $L \subset \mathbf{A}^n$ be a line through the origin, given parametrically as $\{ta : t \in k\}$ for some fixed point a distinct from the origin. If the ideal of X is generated by $\{f_1, \ldots, f_r\}$, then the intersection $X \cap L$ is defined by

$$f_1(ta) = \cdots = f_r(ta) = 0.$$

This is a system of polynomials in the single variable t (picking out a structured algebraic set on the line $L \approx \mathbf{A}^1$). So, we can replace the set of polynomials by its greatest common divisor

$$f(t) = gcd(f_1(ta), \ldots, f_r(ta)) = c \prod (t - \alpha_i)^{n_i}.$$

Definition 3.2.10. Define the *intersection multiplicity* of X and L at 0 to be the multiplicity of the factor t in the polynomial $f(t)$. Since we started with a line that passed through the origin, the multiplicity must be at least 1. We will say that L *touches* X *at* 0 if the intersection multiplicty is at least 2.

Theorem 3.2.11: *The Zariski tangent space of X at 0 is the union of the lines that touch X at 0.*

Proof: Write

$$dF = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(x) t_i.$$

It is easy to check that

$$d(F + G) = dF + dG$$

and

$$d(FG) = F(x)d(G) + G(x)dF.$$

Now write $I(X) = (F_1, \ldots, F_r)$. Then the tangent space to X at x is defined by the equations

$$dF_1 = \cdots = dF_r = 0.$$

To see this, let $M = (x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$ and let m be the image of this ideal in $A(X)$. Then there is an exact sequence

$$I(X) \to M/M^2 \to m/m^2 \to 0.$$

By duality, the tangent space to X at the origin is the subspace to the tangent space to $\mathbf{A}^n$ at the origin consisting of those linear forms $\Theta = \sum t_i D_i$ that vanish on functions in $I(X)$. However, the differential dF is just $\Theta(F)$ under the natural pairing; the claim follows.

Now let $L = \{ta : t \in k\}$ be a (parametric) line through the origin. Write

$$F_j(x) = \sum_{i=1}^{n} a_i x_i + h.o.t. = dF_j + G_j.$$

Then

$$F_j(at) = (\sum_{i=1}^{n} a_i a)t + t^2 * stuff.$$

So, t^2 divides $F_j(ta)$ if and only if $dF_j(a) = 0$. In other words, the line L touches X at 0 if and only if the point a defining the line lies in the tangent space to X at 0. ∎

Definition 3.2.12. Let X be a structured algebraic set in $\mathbf{A}^n$ defined by the ideal $I(X) = (F_1, \ldots, F_r)$. Let $x \in X$ be any point. The *Jacobian matrix* of X at x is defined to be $J_X(x) = ((\partial F_i / \partial x_j)(x))$.

Lemma 3.2.13: *Let $X \subset \mathbf{A}^n$ and let $x \in X$. If m_x is the maximal ideal in $A(X)$ of a point $x \in X$, then*

$$dim(m_x/m_x^2) = n - rank(J_X(x)).$$

Proof: We have just seen that the tangent space to X at x is the null space of the Jacobian matrix. Since the tangent space is dual to m_x/m_x^2, the result follows. ∎

Example 3.2.14. Let $X = Z(F) \subset \mathbf{A}^n$ be a hypersurface. Then the tangent space to X at a point $x = (x_1, \ldots, x_n)$ is defined by the single linear equation

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) t_i = 0.$$

Thus, the tangent space is either $\mathbf{A}^{n-1}$ (if at least one coefficient is nonzero) or $\mathbf{A}^n$ (if all the coefficients xs vanish). One can show that if X is a variety, then there exists at least one point (and hence an entire open set) where the tangent space equals $\mathbf{A}^{n-1}$.

Definition 3.2.15. Let $X \subset \mathbf{A}^n$ be an affine variety of dimension d. A point $x \in X$ is called *nonsingular* if $rank(J_X(x)) = n - d$.

Definition 3.2.16. Let A be a local ring with maximal ideal M. We say that A is *regular* if $dim(M/M^2) = dim(A)$.

Proposition 3.2.17: *Let $X \subset \mathbf{A}^n$ be an affine variety and let $x \in X$. Then x is a nonsingular point of X if and only if the local ring $\mathcal{O}_{X,x}$ is regular.*

Proof: Let $M \subset A(X)$ be the maximal ideal of x, and let $m \subset \mathcal{O}_{X,x}$ be the maximal ideal in the local ring. We already know that

$$m/m^2 \approx M/M^2 \approx \Theta^*_{X,x}$$

and $\Theta_{X,x}$ is isomorphic to the null space of the Jacobian. So,

$$dim(m/m^2) + rank(J_X(x)) = n.$$

One also has $A(X) \subset \mathcal{O}_{X,x} \subset k(X)$. Thus, all three rings have the same transcendence degree $d = dim(A(X)) = dim(\mathcal{O}_{X,x})$. So, $\mathcal{O}_{X,x}$ is regular if and only if $d = dim(m/m^2)$ if and only if $rank(J_X(x)) = n - d$ if and only if x is a nonsingular point on X. ∎

Theorem 3.2.18: *Let X be a variety. Then the set of singular points in X is a proper closed subset.*

Proof: We first show that the set of singular points is closed. This is a local property. Since the affine open subsets form a basis for the topology, we can assume $X \in \mathbf{A}^n$ is an affine variety of dimension d. The proof of the previous result shows that $rank(J_X(x)) \leq n-d$ for all points $x \in X$. Thus, the set of singular points is the set where the rank is strictly less than $n-d$. So, the set of singular points is the algebraic set defined by the ideal generated by $I(X)$ together with the determinants of all $(n-d) \times (n-d)$ submatrices of the Jacobian.

Next, we show that the set of singular points is proper. Since birationally isomorphic varieties have isomorphic open subsets, this property is birational. So, we can assume that $X = Z(F)$ is an irreducible hypersurface in $\mathbf{A}^n$. If the singular set is all of X, then one has an equality of ideals $(F) = I(X) = (F, \partial F/\partial x_1, \ldots, \partial F/\partial x_n)$. But the partial derivatives have degree smaller than the degree of F. The equality of ideals can only hold if all the partials vanish. In characteristic 0, this can only happen if F is a constant. In characteristic p, it can only happen if F is a p-th power. Both cases are impossible, since X is an irreducible hypersurface. ∎

Example 3.2.19.

(i) $y^2 = x^3 + x^2$ has a singularity at $(0,0)$, and no other singular points.

(ii) $y^2 = x^3$ has a singularity at $(0,0)$, and no other singular points. Even though the tangent space is the same as that of the previous case, we want to think of these as different kinds of singularities; we need better invariants to do so.

(iii) $(x^2+y^2)^2 + 3x^2y - y^3$ is a three-leaved rose. It has a singularity at $(0,0)$, and no other singular points. This singularity is a triple point.

Definition 3.2.20. Let A be a (local) ring with (maximal) ideal M. The *completion* of A along M is defined to be the universal object that maps compatibly onto the natural sequence

$$\cdots \to A/M^{i+1} \to A/M^i \to A/M^{i-1} \to \cdots A/M.$$

The completion will be denoted by $\hat{I}$. The usual category theoretic notation is to write

$$\hat{I} = \lim_{\leftarrow} A/M^i;$$

this is an example of an *inverse limit* .

Remark 3.2.21. We can put a topology on A by taking the M^i to be the open neighborhood of 0, and making the topology translation invariant. Use Cauchy sequences to complete A in the topological sense; the result is naturally a ring, and is canonically isomorphic to the completion of A along M. It is clear either from this construction or from the universal property that the completion comes equipped with a natural map $A \to \hat{I}$.

Example 3.2.22.

(i) Let $A = k[x]$ and $M = (x)$. Then $\hat{I} = k[[x]]$ is the ring of formal power series. The natural map $k[x] \to k[[x]]$ from the universal property is the usual map identifying a polynomial with a (finite) power series.

(ii) In the same way, the completion of $k[x_1, \ldots, x_n]$ at the maximal ideal of the origin can be identified with the ring $k[[x_1, \ldots, x_n]]$ of formal power series in n variables.

(iii) Let $A = \mathbf{Z}$ and let $M = (p)$ be the ideal generated by a prime p. Then $\hat{I} = \mathbf{Z}_p$ is usually called the ring of p-adic integers. Elements of $\mathbf{Z}_p$ can be thought of as "power series" in the variable p. More precisely, any element of $\mathbf{Z}_p$ can be written uniquely in the form

$$\sum_{i=0}^{\infty} a_i p^i,$$

where $a_i \in \{0, 1, \ldots, p-1\}$. Actual integers are identified with their expansions in base p. You add and multiply using the usual formulas; the only essential difference is that the sum or product of two "monomials" of the same degree may not be another monomial. (This occurs because $a_i + b_j$ or $a_i b_j$ can be bigger than p.)

We would like to try to generalize these examples, in order to think of elements in completions as power series, but in as coordinate-free a way as possible. The basic tool form commutative algebra is the following.

Theorem 3.2.23: *(Nakayama's Lemma) Let N be a module of finite type over a local ring A with maximal ideal M. If $u_1, \ldots, u_n \in N$ generate N/MN, then they also generate N.*

Proof: Let S denote the submodule of N generated by the u_i. Thus, the hypotheses force $(N/S)/M(N/S) = 0$. Since it suffices to show that $(N/S) = 0$, we can reduce to the case that $N/MN = 0$, where we need to show that $N = 0$.

Let $w_1, \ldots, w_s$ be generators for N. If $s = 1$, then

$$N = MN = M * Aw_s = M * w_s.$$

In particular, we can write $w_s = mw_s$ for some element $m \in M$. Hence, $(1-m)w_s = 0$. Since $1 - m \notin M$, it must be a unit in the local ring A. Therefore, $w_s = 0$, and $N = 0$.

Now suppose $s > 1$, and use induction. Write $N' = N/Aw_s$. Then N' is generated by $s-1$ elements, so $N' = 0$. Thus, $N = Aw_s$ is generated by one element, and we're done. ∎

Definition 3.2.24. We say that a set of elements $\{u_1, \ldots, u_n\} \subset \mathcal{O}_{X,x}$ forms a *system of local parameters* at the point x if their images form a basis of M_x/M_x^2.

Corollary 3.2.25: *Let X be an algebraic set, and let $\{u_1, \ldots, u_n\}$ be a system of local parameters at a point $x \in X$. Then the homomorphism $\varphi : k[[x_1, \ldots, x_n]] \to \hat{\mathcal{O}}_{X,x}$ that sends $x_i \mapsto u_i$ is a surjection.*

Proof: By Nakayama's Lemma, the u_i generate the maximal ideal, and the monomials in the u_i generate the powers of the maximal ideal. ∎

Theorem 3.2.26: $\hat{\mathcal{O}}_{X,x} \approx k[[u_1,\ldots,u_n]]$ *if and only if x is a nonsingular point of X.*

Proof: By the previous result, φ is surjective. Now the point x is nonsingular if and only if $dim(M/M^2) = dim(\mathcal{O}_{X,x}) = dim(\hat{\mathcal{O}}_{X,x})$. However, if φ had a kernel, the dimension would decrease. Thus, φ is also injective. ∎

Definition 3.2.27. Let $P \in X$ and $Q \in Y$ be points on algebraic sets. We say that P and Q are *analytically isomrphic* if there is an isomorphism $\hat{\mathcal{O}}_{X,P} \approx \hat{\mathcal{O}}_{Y,Q}$.

Example 3.2.28.

(i) If P and Q are analytically isomorphic points on algebraic varieties X and Y, then $dim(X) = dim(Y)$.

(ii) If P and Q are nonsingular points on two varieties of the same dimension, then P and Q are analytically isomorphic.

(iii) Let P be the origin on the nodal cubic curve $y^2 = x^3 + x^2$, and let Q be the origin on the reducible curve $xy = 0$ consisting of the two coordinate axes. Then P and Q are analytically isomorphic. To prove this, we need to exhibit an isomorphism between the two rings
$$\hat{\mathcal{O}}_P = k[[x,y]]/(y^2 - x^2 - x^3) \text{and } \hat{\mathcal{O}}_Q = k[[x,y]]/(xy)$$
So, we have to find power series $g, h \in k[[x,y]]$ such that
$$\begin{aligned} g &= (y+x) + g_2 + g_3 + \ldots \\ h &= (y-x) + h_2 + h_3 + \ldots \\ gh &= y^2 - x^2 - x^3. \end{aligned}$$
We start by writing
$$gh = y^2 - x^2 + g_2(y-x) + h_2(y+x) + (degree \geq 3).$$
So,
$$g_2(y-x) + h_2(y+x) = x^3.$$
We can take $g_2 = xy + \frac{1}{2}y^2$ and $h_2 = x^2 - \frac{1}{2}y^2$. Next, we need to solve
$$(y-x)g_3 + (y+x)h_3 = -g_2h_2.$$
We can always find such a solution (because $y - x$ and $(y + x)$ generate the maximal ideal). Continue until you build the entire power series. In this example, it is interesting to note that the ring $\mathcal{O}_P$ is an integral domain, but its completion $\hat{\mathcal{O}}_P$ is not.

(iv) Let $X \subset \mathbf{A}^2$ be an irreducible plane curve and let $P = (0,0)$ be a point of X. The irreducible function cutting out X has the form
$$f = f_r + f_{r+1} + \cdots,$$
where each f_i is a homogeneous polynomial of degree i. In this case, we say that P is an *r-fold point* of X. If f_r is a product of r distinct linear factors, then we say that P is *ordinary* . When $r = 1$, the point P is nonsingular. Any two ordinary double points are analytically isomorphic. Any two ordinary triple points are analytically isomorphic. However, there is a one-parameter family of different ordinary fourfold points!